Last time:

introduced covariant derivative

$$\partial_\mu \phi^\ell \longmapsto D_\mu \phi^\ell = (\partial_\mu - i q_\ell A_\mu)\phi^\ell$$

In the case of the Dirac field, the above gives

$$S(\psi, \bar\psi, A) = \bar\psi\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu$$

Note: above action is only gauge inv. for zero photon mass $\mu = 0$:

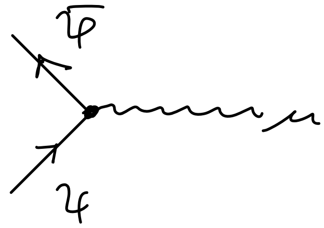
$$S_{\mu=0}(\psi, \bar\psi, A) = S(\psi, \bar\psi) + S(F_{\mu\nu}) + A_\mu \underbrace{e\bar\psi\gamma^\mu\psi}_{= J^\mu(x)}$$

Noether current for $\psi \mapsto e^{ie\theta}\psi$

We keep μ finite here as a regulator for the propagator:

$$\frac{i}{k^2 - \mu^2}\left(\frac{k_\mu k_\nu}{\mu^2} - \eta_{\mu\nu}\right)$$

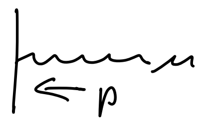
<u>Feynman rules:</u>

- vertex 

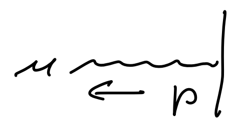
$= ie\gamma^\mu$

- propagator: $\dfrac{i}{k^2-\mu^2}\left(\dfrac{k_\mu k_\nu}{\mu^2}-\eta_{\mu\nu}\right)$

- external photon lines:

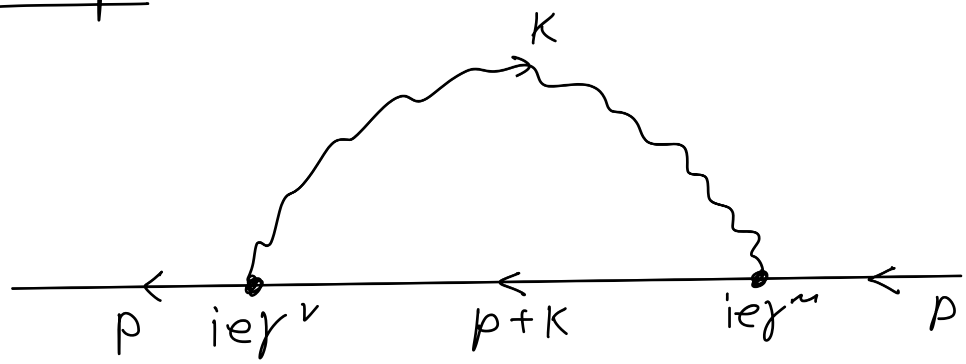
$\overbrace{A_\mu |\vec{p}}\rangle =$ 

$= \varepsilon_\mu(p)$

$\langle \overbrace{\vec{p}| A_\mu} =$ 

$= \varepsilon^*_\mu(p)$

<u>example:</u>



$$= (ie)^2\, i^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2-\mu^2}\left(\frac{k_\mu k_\nu}{\mu^2}-\eta_{\mu\nu}\right)$$

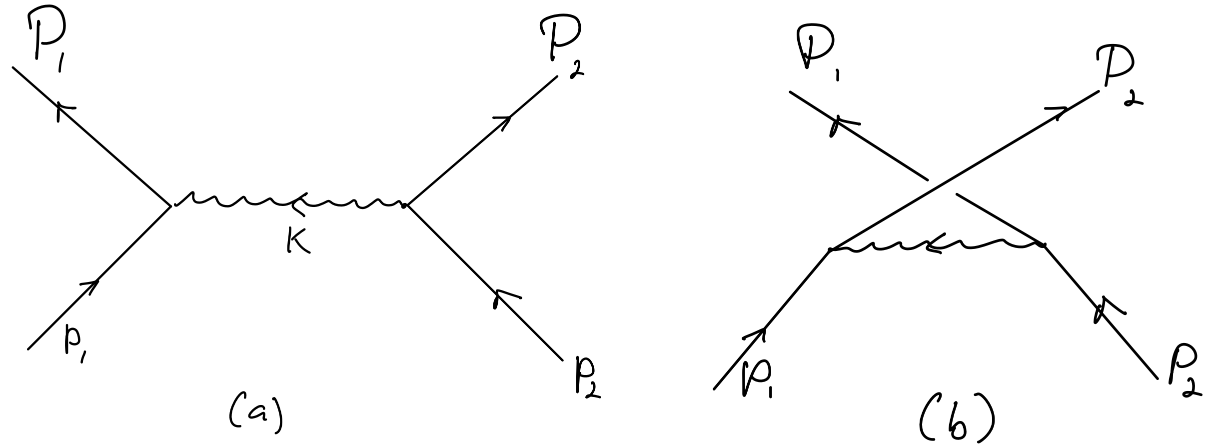
$$\times\ \bar{u}(p)\,\gamma^\nu\, \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\,\gamma^\mu u(p)$$

§3.4 Electron Scattering and Gauge invariance

Electron scattering

Consider two electrons scatter off each other

$\longrightarrow$ to order e^2 :



Applying the Feynman rules, we get for diagram (a):

$$A(P_1, P_2) = (-ie)^2 \frac{i}{(P_1 - p_1)^2 - \mu^2} \left(\frac{K_\mu K_\nu}{\mu^2} - \eta_{\mu\nu} \right)$$

$$\times \bar{u}(P_1) \gamma^\mu u(p_1) \bar{u}(P_2) \gamma^\nu u(p_2)$$

$$= (-ie)^2 \frac{(-i)}{(P_1 - p_1)^2 - \mu^2} \bar{u}(P_1) \gamma^\mu u(p_1) \bar{u}(P_2) \gamma_\mu u(p_2) \qquad (1)$$

where use was made of

$$k_\mu \bar{u}(P_1)\gamma^\mu u(p_1) = (P_1 - p_1)_\mu \bar{u}(P_1)\gamma^\mu u(p_1)$$
$$= \bar{u}(P_1)(\cancel{P}_1 - \cancel{p}_1)u(p_1)$$
$$= \bar{u}(P_1)(m-m)u(p_1) = 0 \quad (*)$$

as $\bar{u}(P_1)$ and $u(p_1)$ satisfy e.o.m.
identity $(*)$ is momentum space version for $\partial_\mu J^\mu(x) = 0$ where $J^\mu = \bar{\psi}\gamma^\mu\psi$

$\longrightarrow$ thus $k_\mu k_\nu / \mu^2$ doesn't enter and we can safely take the limit $\mu \to 0$, giving

$$A(P_1, P_2) = \frac{ie^2}{(P_1 - p_1)^2} \bar{u}(P_1)\gamma^\mu u(p_1)\bar{u}(P_2)\gamma_\mu u(p_2)$$

By Fermi statistics, the amplitude for diagram (b) is $-A(P_2, P_1)$

$\longrightarrow$ thus total amplitude is:

$$\mathcal{M} = A(P_1, P_2) - A(P_2, P_1) \quad (2)$$

The "cross section" (likelihood) is the square of $\mathcal{M}$:

$$|\mathcal{M}|^2 = \left[|A(P_1, P_2)|^2 + (P_1 \leftrightarrow P_2) \right]$$
$$- 2\mathrm{Re}\, A(P_2, P_1)^* A(P_1, P_2) \qquad (3)$$

the first term is :

$$|A(P_1, P_2)|^2 = \frac{e^4}{(P_1 - P_2)^4} \left[\bar{u}(P_1) \gamma^\mu u(p_1) \bar{u}(p_1) \gamma^\nu u(P_1) \right]$$
$$\times \left[\bar{u}(P_2) \gamma_\mu u(p_2) \bar{u}(p_2) \gamma_\nu u(P_2) \right] \qquad (4)$$

In simplest experiments, initial electrons are unpolarized and final polarization is not measured

$\longrightarrow$ have to take average:

$$\sum_s u(p,s) \bar{u}(p,s) = \frac{\not{p} + m}{2m} \qquad (**)$$

Define

$$\tau^{\mu\nu}(P_1, p_1) = \sum_s \sum_S \bar{u}(P_1, S) \gamma^\mu u(p_1 s) \bar{u}(p_1, s) \gamma^\nu u(P_1, S)$$
$$= \frac{1}{(2m)^2} \mathrm{tr}\left[(\not{P}_1 + m) \gamma^\mu (\not{p}_1 + m) \gamma^\nu \right]$$

$$(4) \longrightarrow |A(P_1, P_2)|^2 = \frac{e^4}{(P_1 - P_2)^4} \tau^{\mu\nu}(P_1, p_1) \tau_{\mu\nu}(P_2, p_2)$$

Similarly, in averaging and summing $A(P_2, P_1)^* A(P_1, P_2)$ we get:

$$K := \sum_{\text{spins}} \bar{u}(P_1)\gamma^\mu u(p_1)\bar{u}(P_2)\gamma_\mu u(p_2)\bar{u}(p_1)\gamma^\nu u(P_2)$$
$$\times \bar{u}(p_2)\gamma_\nu u(P_1)$$

applying (6), one can again write K as a trace.

Let's evaluate the traces:

$$\tau^{\mu\nu}(P_1, p_1) = \frac{1}{(2m)^2}\left[tr(\not{P_1}\gamma^\mu \not{p_1}\gamma^\nu) + m^2 tr(\gamma^\mu\gamma^\nu)\right]$$

(trace of odd number of γ-matrices vanishes)

Next, write

$$tr(\not{P_1}\gamma^\mu \not{p_1}\gamma^\nu) = P_{1\rho} p_{1\lambda}\, tr(\gamma^\rho\gamma^\mu\gamma^\lambda\gamma^\nu)$$

Now, using

$$tr\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma = 4(\eta^{\mu\nu}\eta^{\lambda\sigma} - \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda})$$

gives

$$\tau^{\mu\nu}(P_1, p_1) = \frac{4}{(2m)^2}\left(P_1^\mu p_1^\nu - \eta^{\mu\nu} P_1 \cdot p_1 + P_1^\nu p_1^\mu + m^2\eta^{\mu\nu}\right)$$

Computing in the relativistic limit, where $m \ll p$, we get for K:

$$K = \frac{1}{(2m)^4} \underbrace{\mathrm{tr}\left(\not{P}_1 \gamma^\mu \not{p}_1 \gamma^\nu \not{P}_2 \gamma_\mu \not{p}_2 \gamma_\nu\right)}$$

$$= -2\,\mathrm{tr}\left(\not{P}_1 \gamma^\mu \not{p}_1 \not{p}_2 \gamma_\mu \not{P}_2\right)$$

$$= -32\, p_1 \cdot p_2 \; P_1 \cdot P_2$$

In the same limit ($m \ll p$):

$$\tau^{\mu\nu}(P_1, p_1) = \frac{4}{(2m)^2}\left(P_1^\mu p_1^\nu + P_1^\nu p_1^\mu - \eta^{\mu\nu} P_1 \cdot p_1\right)$$

thus giving in total

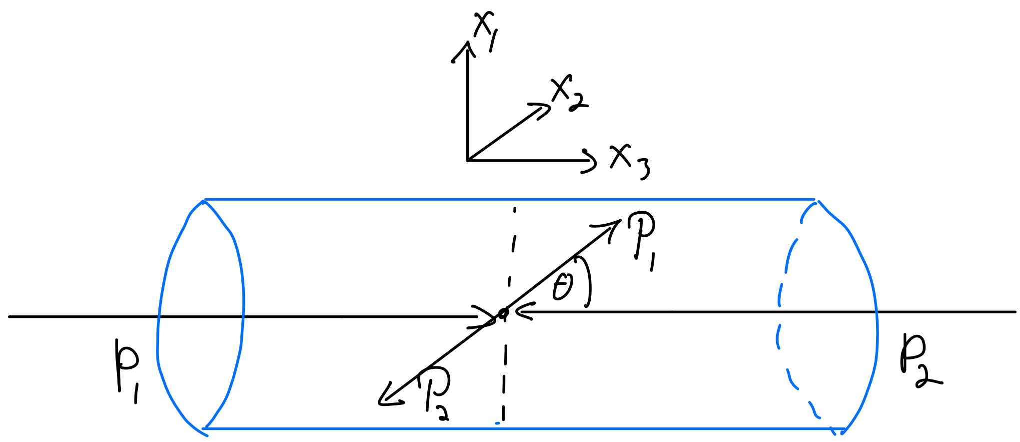
$$\tau^{\mu\nu}(P_1, p_1)\, \tau_{\mu\nu}(P_2, p_2)$$

$$= \frac{16}{(2m)^2}\left(P_1^\mu p_1^\nu + P_1^\nu p_1^\mu - \eta^{\mu\nu} P_1 \cdot p_1\right) \times \left(2 P_2^\mu p_2^\nu - \eta^{\mu\nu} P_2 \cdot p_2\right)$$

$$= \frac{16 \cdot 2}{(2m)^2}\left(p_1 \cdot p_2 \; P_1 \cdot P_2 + p_1 \cdot P_2 \; p_2 \cdot P_1\right)$$

In the center-of-mass frame in the relativistic limit:

$$p_1 = E(1, 0, 0, 1),$$
$$p_2 = E(1, 0, 0, -1),$$
$$P_1 = E(1, \sin\theta, 0, \cos\theta),$$
$$P_2 = E(1, -\sin\theta, 0, -\cos\theta)$$



Hence,

$$p_1 \cdot p_2 = P_1 \cdot P_2 = 2E^2,$$
$$p_1 \cdot P_1 = p_2 \cdot P_2 = 2E^2 \sin^2\left(\frac{\theta}{2}\right)$$
$$p_1 \cdot P_2 = p_2 \cdot P_1 = 2E^2 \cos^2(\theta/2)$$
$$(P_1 - p_1)^4 = (-2p_1 \cdot P_1)^2 = 16E^4 \sin^4(\theta/2)$$

Putting everything together, one obtains

$$\frac{1}{2} \sum_s \sum_S |\mathcal{M}|^2 = \frac{e^4}{4m^4} f(\theta),$$

where

$$f(\theta) = \frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)}$$

$$+ \frac{1+\sin^4(\theta/4)}{\cos^4(\theta/2)}$$

- the first term favors forward scattering (photon propagator blows up at $K \sim 0$)
- the last term ensures the symmetry $\theta \longmapsto \pi - \theta$ (electrons are indistinguishable)
- the second term is a quantum interference